E-ISSN: 2395-1702 P-ISSN: 2395-0382
Volume 01- Issue 04-, PP-19-25

Research Paper

THE FORCING TOTAL EDGE DOMINATION NUMBER OF A GRAPH

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Abstract:

Let G be a connected graph and S a minimum total edge dominating set of G. A subset$T \subseteq S$is called aforcing subset for S if S is the unique minimum total edge dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. Theforcing total edge domination number of S, denoted by$f_{\gamma_{te}}(S)$, is the cardinality of a minimum forcing subset of S. The forcing total edge domination number of G, denoted by$f_{\gamma_{te}}(G)$, is$f_{\gamma_{te}}(G) = \min\left\{f_{\gamma_{te}}(S)\right\}$, where the minimum is taken over all minimum total edge dominating sets S in G. Some general properties satisfied by this concept are studied. Connected graphs with forcing total edge domination number 0 or 1 are characterized. Some realization results are given.

Keywords: total edge domination number, forcing edge domination number, forcing total edge domination number.
Mathematics subject classification: 05C69
Field: Graph Theory; **Subfield:** Domination

1. Introduction

All graphs under our consideration are finite, undirected, without loops, multiple edges and isolated vertices. Terms not defined here are used in the sense of Harary [3].A concept of edge domination was introduced by Mitchell and Hedetniemi [4].An edge dominating set S of G is called a total edge dominating set of G if $\langle S \rangle$ has no isolated edges. The total edge domination number$\gamma_{te}(G)$of Gis the minimum cardinality taken over all total edge dominating sets of G.

We also introduce the concept of the forcing total edge domination number $f_{\gamma_{te}}(G)$of a connected graph G with at least 3 vertices. Let G be a connected graph and S a minimum total edge dominating set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum total edge dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing total edge domination number of S, denoted by $f_{\gamma_{te}}(S)$, is the cardinality of a minimum forcing subset of S. The forcing total edge domination number of G, denoted by $f_{\gamma_{te}}(G)$, is $f_{\gamma_{te}}(G) = \min\{f_{\gamma_{te}}(S)\}$, where the minimum is taken over all minimum total edge dominating sets S in G. For forcing domination number we refer to [1].

Definition 1.1

Let G be a connected graph and S a minimum total edge dominating set of G. A subsetT $\subseteq$ Sis called aforcing subset for S if S is the unique minimum total edge dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subsetof S. Theforcing total edge domination number of S, denoted by$f_{\gamma_{te}}(S)$, is the cardinality of a minimum forcing subset of S. The forcingtotal edge domination number of G, denoted by$f_{\gamma_{te}}(G)$, is$f_{\gamma_{te}}(G) = \min\{f_{\gamma_{te}}(S)\}$, where the minimum is taken over all minimum total edge dominating sets S in G.

Example 1.2

For the graph G given in Figure 1,$S = \{v_4v_5, v_5v_2\}$is the unique minimum total edge dominating set of G so that$f_{\gamma_{te}}(G) = 0$ and for the graph G given in Figure 2,$S_1 = \{v_3v_5, v_2v_3, v_3v_4\}$, $S_2 = \{v_3v_5, v_2v_3, v_1v_2\}$ and $S_3 = \{v_3v_5, v_3v_4, v_1v_4\}$ are the only three minimum total edge dominating sets of G such that$f_{\gamma_{te}}(S_1) = 2$ and $f_{\gamma_{te}}(S_2) = f_{\gamma_{te}}(S_3) = 1$so that$f_{\gamma_{te}}(G) = 1$.

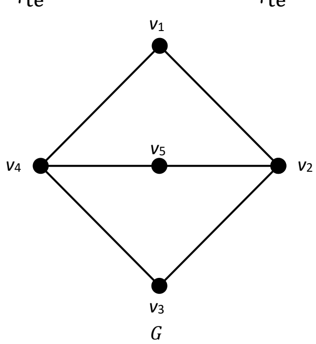

Figure 1

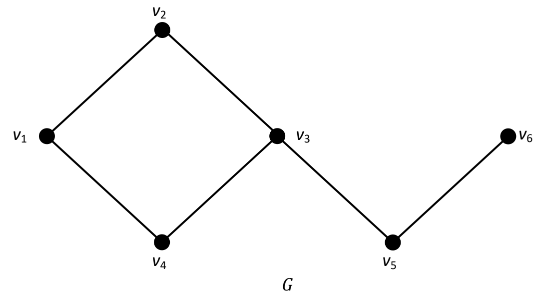

Figure 2

The next theorem follows immediately from the definition of the total edge domination number and the forcing total edge domination number of a connected graph G.

Theorem 1.3

For every connected graph G,$0 \leq f_{\gamma_{te}}(G) \leq \gamma_{te}(G)$.

Remark 1.4

The bounds in Theorem 1.3 are sharp. For the graphG given in Figure 1,$f_{\gamma_{te}}(G) = 0$and for the graph$G = K_n$, $f_{\gamma_{te}}(G) = \gamma_{te}(G) = 2$. Also, all the inequalities in Theorem1.3 are strict. For the graphGgiven in Figure 2,$f_{\gamma_{te}}(G) = 1$and $\gamma_{te}(G) = 3$. Thus$0 < f_{\gamma_{te}}(G) < \gamma_{te}(G)$.

Theorem 1.5

LetG be a connected graph. Then

(a) $f_{\gamma_{te}}(G) = 0$if and only if G has a unique minimum total edge dominating set.

(b) $f_{\gamma_{te}}(G) = 1$if and only ifGhas at least two minimum total edge dominating sets, one of which is a unique minimum total edge dominating set containing one of its elements, and

(c) $f_{\gamma_{te}}(G) = \gamma_{te}(G)$if and only if no minimum total edge dominating set ofG is the unique minimum total edge dominating set containing any of its proper subsets.

Proof

(a) Let$f_{\gamma_{te}}(G) = 0$. Then, by definition,$f_{\gamma_{te}}(S) = 0$for some minimum total edge dominating set S ofG so that the empty set $\phi$is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that Sis the unique minimum total edge dominating set of G. The converse is clear.

(b) Let $f_{\gamma_{te}}(G) = 1$. Then by part (a),G has at least two minimum total edge dominating sets. Also, since$f_{\gamma_{te}}(G) = 1$, there is a singleton subset T of a minimum total edge dominating set S of G such that Tis not a subset of any other minimum total edge dominating set ofG. Thus Sis the unique minimum total edge dominating set containing one of its elements. The converse is clear.

(c) Let$f_{\gamma_{te}}(G) = \gamma_{te}(G)$. Then$f_{\gamma_{te}}(S) = \gamma_{te}(G)$for every minimum total edge dominating set SinG. Since $m \geq 2$, $\gamma_{te}(G) \geq 2$ and hence$f_{\gamma_{te}}(G) \geq 2$. Then by part (a), Ghas at least two minimum total edge dominating sets and so the empty set $\phi$is not a forcing subset for any minimum total edge dominating set ofG. Since$f_{\gamma_{te}}(S) = \gamma_{te}(G)$, no proper subset of S is a forcing subset of S. Thus no minimum total edge dominating set ofG is the unique minimum total edge dominating set containing any of its proper subsets. Conversely, the data implies thatG contains more than one minimum total edge dominating set and no subset of any minimum total edge dominating sets S other than S is a forcing subset for S. Hence it follows that$f_{\gamma_{te}}(G) = \gamma_{te}(G)$.

Definition 1.6

An edge e of a connected graph G is said to be a total edge dominating edge of G if e belongs to every minimum total edge dominating set of G. If G has a unique minimum total edge dominating set S, then every edge of S is a total edge dominating edge of G.

Example 1.7

For the graph G given in Figure 1, $S = \{v_4v_5, v_5v_2\}$is the unique minimum total edge dominating set of G so that both the edges in S are total edge dominating edges of G. For the graph G given in Figure 2, an edge v_3v_5 belongs to every minimum total edge dominating set of G. Therefore v_3v_5 is the unique total edge dominating edge of G.

Theorem 1.8

Let G be a connected graph and let $\mathfrak{I}$ be the set of relative complements of the minimum forcing subsets in their respective minimum total edge dominating sets in G. Then $\bigcap_{F\in\mathfrak{I}} F$ is the set of total edge dominating edges of G.

Corollary 1.9

Let G be a connected graph and S a minimum total edge dominating set of G. Then no total edge dominating edge of G belongs to any minimum forcing set of S.

Theorem 1.10

Let G be a connected graph andX be the set of all total edge dominating edges of G. Then $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X|$.

Remark 1.11

The bound in Theorem 1.10 is sharp. For the graph G given in Figure 1, $\gamma_{te}(G) = 2$, $|X| = 2$, $f_{\gamma_{te}}(G) = 0$ and $\gamma_{te}(G) - |X| = 0$ so that $f_{\gamma_{te}}(G) = \gamma_t(G) - |X|$. Also the bound in Theorem 1.10 is strict. For the graph G given in Figure 2, $\gamma_{te}(G) = 3$, $|X| = 1$, $f_{\gamma_{te}}(G) = 1$ and $\gamma_{te}(G) - |X| = 2$ so that $f_{\gamma_t}(G) < \gamma_t(G) - |W|$.

In the following we determine the forcing total edge domination number of some standard graphs.

Theorem 1.12

For any graph $G = P_n (n \geq 3)$, $f_{\gamma_{te}}(G) = \begin{cases} 0 & \text{if } n \equiv 1 (\text{mod } 4) \text{and } n \neq 3 \\ 2 & \text{if } n \equiv 3 (\text{mod } 4) \\ 1 & \text{if } n \text{ is even and } n \neq 6 \end{cases}$

Proof

Let$E(P_n)$be $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

Case 1.nis odd.

Subcase i. Let$n = 3$.

Then $S = \{v_1v_2, v_2v_3\}$ is the unique minimum total edge dominating set of G, so that $f_{\gamma_{te}}(G) = 0$.

Subcase ii. Let $n \equiv 3 (\mathrm{mod}\ 4)$.

Let $n = 4k+3,\ k \geq 1$. Let S be any γ_{te}-set of G. Then it is easily verified that any singleton subset of S is a subset of another γ_{te}-set of G and so $f_{\gamma_{te}}(G) \geq 1$. Now $s_1 = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \dots, v_{4k+1}v_{4k+2}, v_{4k+2}v_{4k+3}\}$ is a γ_{te}-set of G. S_1 is the unique γ_{te}-set of G containing $\{v_1v_2, v_{4k+2}v_{4k+3}\}$ so that $f_{\gamma_{te}}(G) = 2$.

Subcase iii. Let $n \equiv 1 (\mathrm{mod}\ 4)$.

Let $n = 4k+1,\ k \geq 1$. Then $S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, \dots, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\}$ is the unique minimum total edge dominating set of G, so that $f_{\gamma_{te}}(G) = 0$.

Case 2. n is even.

Subcase i. Let $n = 6$.

Then $S = \{v_2v_3, v_3v_4, v_4v_5\}$ is the unique γ_{te}-set of G, so that $f_{\gamma_{te}}(G) = 0$.

Subcase ii. Let $n \equiv 0 (\mathrm{mod}\ 4)$.

Let $n = 4k,\ k \geq 1$. Then $S = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, \dots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\}$ is the unique γ_{te}-set of G containing $\{v_1v_2\}$, so that $f_{\gamma_{te}}(G) = 1$.

Subcase iii. Let $n \equiv 2 (\mathrm{mod}\ 4)$.

Let $n = 4k+2,\ k \geq 2$. Then $S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, \dots, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\}$ is the unique γ_{te}-set of G containing $\{v_{4k-2}v_{4k-1}\}$ so that $f_{\gamma_{te}}(G) = 1$.

■

Theorem 1.13

For any graph $G = C_n, (n \geq 3)$, $f_{\gamma_{te}}(G) = \begin{cases} 4 & \text{if } n \equiv 2 (\mathrm{mod}\ 4) \\ 2 & \text{otherwise} \end{cases}$

Proof

Let C_n be $v_1, v_2, \dots, v_n, v_1$.

Case 1. n is odd.

Subcase i. Let $n+1 \equiv 0 (\mathrm{mod}\ 4)$.

Let $n = 4k-1,\ k \geq 1$. Let S be any γ_{te}-set of G. Then it is easily verified that any singleton subset of S is a subset of another γ_{te}-set of G and so $f_{\gamma_{te}}(G) \geq 1$. Now $S_1 = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \dots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\}$ is the unique γ_{te}-set of G containing $\{v_1v_2, v_{4k-2}v_{4k-1}\}$ so that $f_{\gamma_{te}}(G) = 2$.

Subcase ii. Let $n-1 \equiv 0 (\mathrm{mod}\ 4)$.

Let $n = 4k+1,\ k \geq 1$. Let S be any γ_{te}-set of G. Then it is easily verified that any singleton subset of S is a subset of another γ_{te}-set of G and so $f_{\gamma_{te}}(G) \geq 1$. Now $s_1 = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \dots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}\}$ is the unique γ_{te}-set of G containing $\{v_{4k-3}v_{4k-2}, v_{4k-1}v_{4k}\}$ so that $f_{\gamma_{te}}(G) = 2$.

Case 2. n is even.

Subcase i. Let $n \equiv 0 (\mathrm{mod}\ 4)$.

Let $n = 4k,\ k \geq 1$. Let S be any γ_{te}-set of G. Then it is easily verified that any singleton subset of S is a subset of another γ_{te}-set of G and so $f_{\gamma_{te}}(G) \geq 1$. Now $s_1 = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \dots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\}$ is the unique γ_{te}-set of G containing $\{v_1v_2, v_2v_3\}$ so that $f_{\gamma_{te}}(G) = 2$.

Subcase ii. Let $n \equiv 2 (\mathrm{mod}\ 4)$.

Let $n = 4k+2,\ k \geq 1$. Let S be any γ_{te}-set of G. Then it is easily verified that any one element or two element or three element subset of S is a subset of another γ_{te}-set of G. Now $S_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_7v_8, v_8v_9, v_{11}v_{12}, v_{12}v_{13}, \dots, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\}$ is the unique γ_{te}-set of G containing $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ so that $f_{\gamma_{te}}(G) = 4$. ■

Theorem 1.14

For the complete graph $G = K_n (n \geq 3)$, $f_{\gamma_{te}}(G) = 2$.

Proof

Since $n \geq 3$, there exists at least two γ_{te}-sets of G so that $f_{\gamma_{te}}(G) \geq 1$. Let S be any γ_{te}-set of G such that $|S| = 2$. It is easily verified that any singleton subset of S is a subset of another γ_{te}-set of G, so that $f_{\gamma_{te}}(G) = 2$.

Theorem 1.15[2]
Let G be a connected graph and W be the set of all edge dominating edges of G. Then $f_{\gamma_e}(G) \le \gamma_e(G) - |W|$.
In the following the forcing edge domination number and the forcing total edge domination number of a graphG are related.
Theorem 1.16
For any integer $a \ge 2$, there exists a connected graph G such that $f_{\gamma_{te}}(G) = f_{\gamma_e}(G) = a$.
Proof
Let $P: x, y$ and $P_i: u_i, v_i$ $(1 \le i \le a)$ be paths of order 2. Let G be a graph obtained from P_i $(1 \le i \le a)$ and P by joining x with each u_i $(1 \le i \le a)$ and y with each v_i $(1 \le i \le a)$. The graph G is shown in Figure 3.

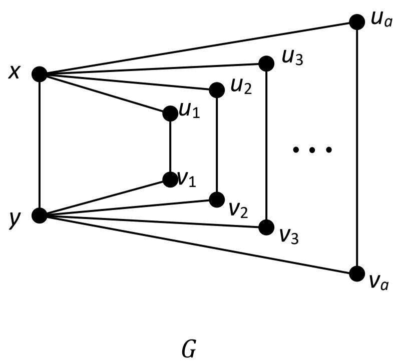

Figure 3

First we show that $\gamma_e(G) = a + 1$. It is easily observed that an edge xy belongs to every minimum edge dominating set of Gand so $\gamma_e(G) \ge 1$. Let $H_i = \{xu_i, u_iv_i, yv_i\}$ $(1 \le i \le a)$. Also it is easily seen that every edge dominating set of Gcontains at least one edge of H_i $(1 \le i \le a)$ and so $\gamma_e(G) \ge a + 1$. Now $S = \{xy\} \cup \{u_1v_1, u_2v_2, \ldots, u_av_a\}$ is an edge dominating set of G so that $\gamma_e(G) = a + 1$.

Next we show that $f_{\gamma_e}(G) = a$. By Theorem1.15, $f_{\gamma_e}(G) \le \gamma_e(G) - \{xy\} = a + 1 - 1 = a$. Now since $\gamma_e(G) = a + 1$ and every minimum edge dominating set of Gcontains $\{xy\}$, it is easily seen that every γ_e-set of Gis of the form $S = \{xy\} \cup \{p_1q_1, p_2q_2, \ldots, p_aq_a\}$, where $p_iq_i \in H_i$ $(1 \le i \le a)$. Let T be any proper subset of Swith $|T| < a$. Then there exists an edge p_jq_j $(1 \le j \le a)$ such that $p_jq_j \notin T$. Let r_js_j be an edge of H_j distinct from p_jq_j. Then $S_1 = \{(S - \{p_jq_j\}) \cup \{r_js_j\}\}$ is a γ_e-set of G properly containing T. Therefore T is not a forcing subset of G. Hence it follows that $f_{\gamma_e}(G) = a$.

Next we claim that $\gamma_{te}(G) = a + 1$. Let $G_i = \{xu_i, yv_i\}$ $(1 \le i \le a)$. It is easily seen that an edge xy belongs to every minimum total edge dominating set of G andso $\gamma_{te}(G) \ge 1$. Also every total edge dominating set of G contains at least one element of G_i $(1 \le i \le a)$ and so $\gamma_{te}(G) \ge a + 1$. Now $S = \{xy\} \cup \{yv_1, yv_2, \ldots, yv_a\}$ is a total edge dominating set of G so that $\gamma_{te}(G) = a + 1$.
Next we show that $f_{\gamma_{te}}(G) = a$. By Theorem1.10, $f_{\gamma_{te}}(G) \le \gamma_{te}(G) - \{xy\} = a + 1 - 1 = a$. Now since $\gamma_{te}(G) = a + 1$ and every minimum total edge dominating set of G contains $\{xy\}$ and at least one edge of G_i $(1 \le i \le a)$, it is easily seen that every γ_{te}-set of Gis of the form $S = \{xy\} \cup \{xc_1, xc_2, \ldots, xc_a\}$, where $xc_i \in G_i$ $(1 \le i \le a)$. Let T be any proper subset of Swith $|T| < a$. Then there exists an edge xc_j $(1 \le j \le a)$ such that $xc_j \notin T$. Let xd_j be an edge of G_j distinct from xc_j. Then $S_1 = \{(S - \{xc_j\}) \cup \{xd_j\}\}$ is a γ_e-set of G properly containing T. Therefore T is not a forcing subset of S. Hence it follows that $f_{\gamma_{te}}(G) = a$.
Theorem1.17

For every pair a, b of integers with $0 \leq a \leq b$, there exists a connected graph G such that $f_{\gamma_{te}}(G) = a$ and $f_{\gamma_e}(G) = b$.

Proof

Let $P: x, y$, $P_i: u_i, v_i (1 \leq i \leq a)$ and $Q_i: r_i, s_i$ $(1 \leq i \leq b-a)$ be paths of order 2. Let H be a graph obtained from P and $P_i (1 \leq i \leq a)$ by joining x with each $u_i (1 \leq i \leq a)$ and y with each $v_i (1 \leq i \leq a)$. Let H' be a graph obtained from Q_i $(1 \leq i \leq b-a)$ by adding new vertex z and joining z with each r_i $(1 \leq i \leq b-a)$. Let G be a graph obtained from H and H' by joining x and z. The graph G is shown in Figure 4.

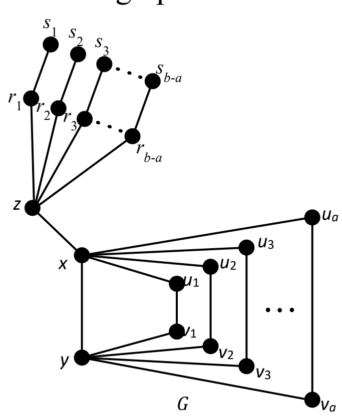

Figure 4

First we claim that $\gamma_e(G) = b + 1$. Let $H_i = \{xu_i, yv_i, u_iv_i\} (1 \leq i \leq a)$ and $R_i = \{zr_i, r_is_i\} (1 \leq i \leq b-a)$. It is easily observed that an edge xy belongs to every minimum edge dominating set of G and so $\gamma_e(G) \geq 1$. Also it is easily seen that every edge dominating set of G contains at least one edge of $H_i (1 \leq i \leq a)$ and at least one edge of $R_i (1 \leq i \leq b-a)$ and so $\gamma_e(G) \geq 1 + a + b - a = b + 1$. Now $S = \{xy\} \cup \{u_1v_1, u_2v_2, \ldots, u_av_a\} \cup \{r_1s_1, r_2s_2, \ldots, r_{b-a}s_{b-a}\}$ is an edge dominating set of G so that $\gamma_e(G) = b + 1$.

Next we show that $f_{\gamma_e}(G) = b$. By Theorem 1.10, $f_{\gamma_e}(G) \leq \gamma_e(G) - \{xy\} = b + 1 - 1 = b$. Since $\gamma_e(G) = b + 1$ and every edge dominating set of G contains $\{xy\}$, it is easily seen that every γ_e-set of G is of the form $S = \{xy\} \cup \{c_1d_1, c_2d_2, \ldots, c_au_a\} \cup \{g_1h_1, g_2h_2, \ldots, g_{b-a}h_{b-a}\}$ where $c_id_i \in H_i$ $(1 \leq i \leq a)$ and $g_ih_i \in R_i$ $(1 \leq i \leq b-a)$. Let T be any proper subset of S with $|T| < b$. Then it is clear that there exists some i and j such that $T \cap H_i \cap R_j = \phi$, which shows that $f_{\gamma_e}(G) = b$.

Next we show that $\gamma_{te}(G) = b + 1$. Let $Z_i = \{xu_i, yv_i\}$ $(1 \leq i \leq a)$ and $X = \{xy, zr_1, zr_2, \ldots, zr_{b-a}\}$. It is easily observed that X is a subset of every minimum total edge dominating set of G and so $\gamma_{te}(G) \geq b - a + 1$. Also it is easily seen that every total edge dominating set of G contains at least one edge of Z_i $(1 \leq i \leq a)$ and so $\gamma_{te}(G) \geq b - a + 1 + a$. Now $S = X \cup \{xu_1, xu_2, \ldots, xu_a\}$ is a total edge dominating set of G so that $\gamma_{te}(G) = b + 1$.

Next we claim that $f_{\gamma_{te}}(G) = a$. By Theorem 1.10, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X| = b + 1 - (b - a + 1) = a$. Now since $\gamma_{te}(G) = b + 1$ and every minimum total edge dominating set of G contains X, it is easily seen that every γ_{te}-set of G is of the form $S = X \cup \{xc_1, xc_2, \ldots, xc_a\}$ where $xc_i \in Z_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exists an edge $xc_j (1 \leq j \leq a)$ such that $xc_j \notin T$. Let xd_j be an edge of Z_j distinct from xc_j. Then $S_1 = \{(S - \{xc_j\}) \cup \{xd_j\}\}$ is a γ_{te}-set of G properly containing T. Therefore T is not a forcing subset of S. This is true for all γ_{te}-sets of G. Hence it follows that $f_{\gamma_{te}}(G) = a$.

Similarly we have proved the following realization results.

Theorem 1.18

For every pair a, b of integers with $0 \leq a \leq b$ there exists a connected graph G such that $f_{\gamma_e}(G) = a$ and $f_{\gamma_{te}}(G) = b$.

Theorem 1.19
For any integer $a \geq 2$, there exists a connected graph G such that $f_{\gamma_{te}}(G) = 0$ and $f_{\gamma_e}(G) = a$.

Theorem 1.20
For any integer $a \geq 2$, there exists a connected graph G such that $f_{\gamma_{te}}(G) = a$ and $f_{\gamma_e}(G) = 0$.

Open Problem 1.21
For every four positive integers a, b, c, d with $2 \leq a \leq b$, $c \geq 0$ and $d \geq 0$, does there exists a connected graph G with $\gamma_e(G) = a$, $\gamma_{te}(G) = b$, $f_{\gamma_e}(G) = c$ and $f_{\gamma_{te}}(G) = d$?

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